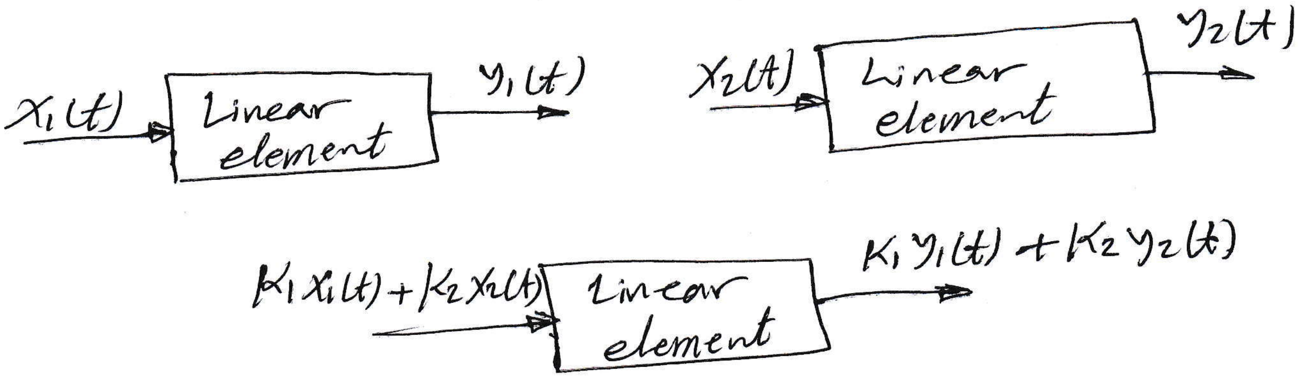
2- Mathematical Modelling :-

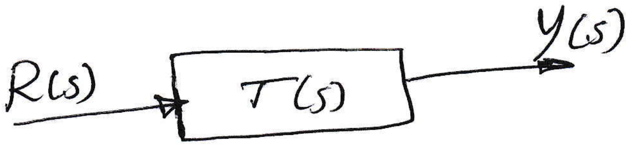
2-1 Mathematical Models of physical Systems :-

The system may be considered to be consisting of an ~~integer~~ inter connection of smaller components or elements, whose behavior can be described by means of mathematical equations or relationship.



2-2 Transfer function :-

It is the ratio of Laplace transform of the output to the Laplace transform of the input.



$$T(s) = \frac{Y(s)}{R(s)}$$

it is simple transfer function

2-2-1 Electrical Systems :-

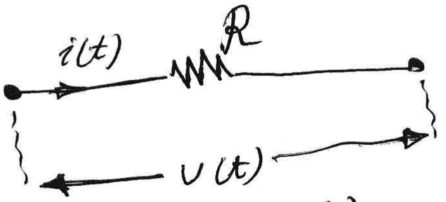
Most of the electrical systems can be modelled by three Basic elements

1- Resistor 2- inductor 3- capacitor.

circuits consisting of these three elements are analysed by using "Kirchoff's" voltage law and current law.

a) Resistor

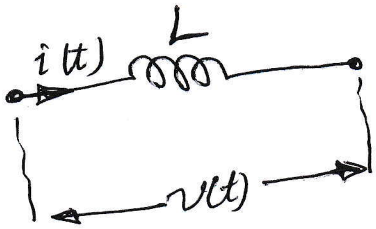
by Ohm's law



$$v(t) = i(t) \cdot R \Rightarrow i(t) = \frac{v(t)}{R}$$

$$i(s) = \frac{v(s)}{R}$$ by Laplace Transform

b) Inductor



by Faraday's law

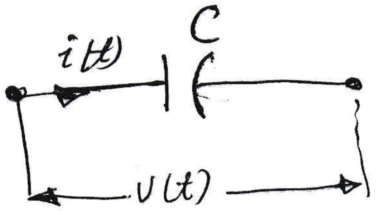
$$v(t) = L \frac{di(t)}{dt}$$

$$i(t) = \frac{1}{L} \int v\, dt$$

or $v(s) = L\, S\, i(s)$ by Lapace transform

$$i(s) = \frac{1}{L} v(s)$$

(C) Capacitor



$$v(t) = \frac{1}{C} \int i \, dt$$

or $$i(t) = C \frac{dv}{dt}$$

$$i(s) = C\, S\, v(s)$$ by Laplace transform

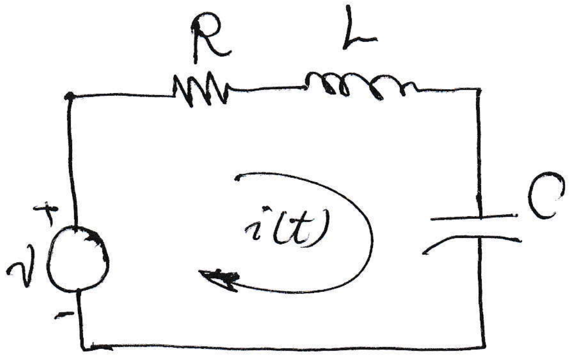
$$v(s) = \frac{1}{CS} \, i(s)$$

example

Consider the network in figure below, obtain the relation between the applied voltage and the current in the form ⓐ Differential equation ⓑ Transfer function.

ⓐ By Kirchhoff's voltage law equation

$$v = R\, i(t) + L \frac{di(t)}{dt} + \frac{1}{C} \int i(t)$$



$$\int i(t)\, dt = q(t)$$

by derivative two sides

$$i(t) = \frac{dq(t)}{dt}$$

$$\therefore \; v = R \frac{dq(t)}{dt} + L \frac{d^2 q(t)}{dt^2} + \frac{q(t)}{C}$$

This is a 2^{nd} order linear differential equation.

b) Transfer function.

Taking Laplace transform, with all initial conditions assumed to be zero, we have

$$LS^2 Q(s) + RS\,Q(s) + \frac{1}{C} Q(s) = V(s)$$

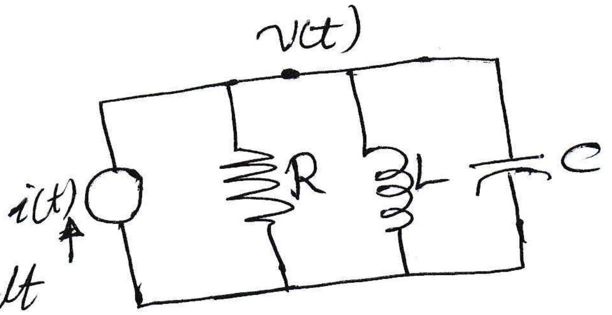
$$\frac{Q(s)}{V(s)} = \frac{1}{Ls + Rs + \frac{1}{C}} = \frac{C}{LCs^2 + RCs + 1}$$

or by $i(s)$

$$V(s) = R\,i(s) + LS\,i(s) + \frac{1}{Cs} i(s)$$

$$\frac{i(s)}{V(s)} = \frac{1}{LS + R + \frac{1}{Cs}} = \frac{Cs}{LCs^2 + RCs + 1}$$

<u>Example 2</u> Consider the parallel RLC network excited by a current source. Find ⓐ differential equation representation ⓑ transfer function representation of the system.



ⓐ by Kirchhoff's current law.

$$i(t) = \frac{v(t)}{R(t)} + \frac{C\,dv(t)}{dt} + \frac{1}{L}\int v\,dt \quad \text{(*)}$$

$\int v\,dt = \psi(t)$ flux linkages

by derivative for both sides

$$v(t) = \frac{d\psi(t)}{dt}$$

$$\frac{1}{R}\frac{d\psi(t)}{dt} + C\frac{d^2\psi(t)}{dt} + \frac{\psi(t)}{L} = i(t)$$

b) by Laplace transform of equation, we have

$$\frac{1}{R}S\,\psi(s) + CS^2\psi(s) + \frac{1}{L}\psi(s) = i(s)$$

$$\frac{\psi(s)}{i(s)} = \frac{1}{CS^2 + \frac{S}{R} + \frac{1}{L}}$$

$$= \frac{1}{\frac{RLCS^2 + \frac{L}{RL}S + \frac{R}{RL}}{RL}} = \frac{RL}{RLCS^2 + \frac{1}{R}S + \frac{1}{L}}$$

if the voltage is taken as the output, taking Laplace transform of eq(*), we get

$$i(s) = \frac{1}{R}v(s) + CS\,v(s) + \frac{1}{LS}v(s)$$

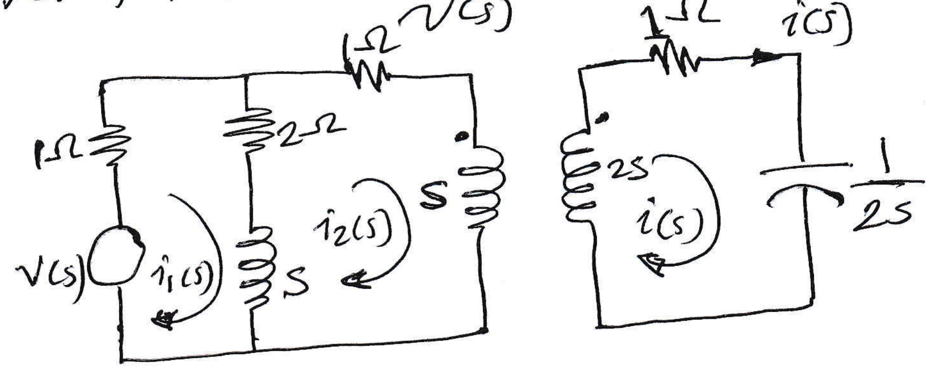
$$i(s) = \left(\frac{1}{R} + Cs + \frac{1}{Ls}\right) v(s)$$

$$i(s) = \frac{LS + RLCS^2 + R}{RLS} v(s)$$

$$\therefore \boxed{\frac{v(s)}{i(s)} = \frac{RLS}{RLCS^2 + LS + R}}$$

Example 3

obtain the transfer function $\frac{I(s)}{V(s)}$ in the network.



$$(3+S)\, I_1(s) - (s+2)\, I_2(s) = v(s) \quad \text{——} (1)$$

$$-(2+S)\, I_1(s) + (3+2S) I_2(s) - S\, I(s) = 0 \quad \text{——} (2)$$

$$-S\, I_2(s) + \left(1 + 2S + \frac{1}{2S}\right) I(s) = 0 \quad \text{——} (3)$$

we have
$$A = \begin{bmatrix} 3+S & -(s+2) & 0 \\ -(2+S) & (3+2S) & -S \\ 0 & -S & \left(1+2S+\frac{1}{2S}\right) \end{bmatrix}, \quad b = \begin{bmatrix} v(s) \\ 0 \\ 0 \end{bmatrix}$$

By Gramer's rule

$$I_1 = \frac{\begin{vmatrix} V(s) & -(s+2) & 0 \\ 0 & (3+2s) & -s \\ 0 & -s & (1+2s+\frac{1}{2s}) \end{vmatrix}}{\begin{vmatrix} 3+s & -(s+2) & 0 \\ -(2+s) & (3+2s) & -s \\ 0 & -s & (1+2s+\frac{1}{2s}) \end{vmatrix}}, \quad I_2 = \frac{\begin{vmatrix} 3+s & V(s) & 0 \\ -(2+s) & 0 & -s \\ 0 & 0 & (1+2s+\frac{1}{2s}) \end{vmatrix}}{\begin{vmatrix} 3+s & -(s+2) & 0 \\ -(s+2) & (3+2s) & -s \\ 0 & -s & (1+2s+\frac{1}{2s}) \end{vmatrix}}$$

$$I = \frac{\begin{vmatrix} 3+s & -(s+2) & V(s) \\ -(2+s) & (3+2s) & 0 \\ 0 & -s & 0 \end{vmatrix}}{\begin{vmatrix} 3+s & -(s+2) & 0 \\ -(2+s) & (3+2s) & -s \\ 0 & -s & (1+2s+\frac{1}{2s}) \end{vmatrix}}$$

for I, we obtain

$$I(s) = \frac{V(s)\left[s(s+2)\right]}{(3+s)\left[(3+2s)\left(1+2s+\frac{1}{2s}\right)-s^2\right]+(s+2)\left[-(s+2)\left(1+2s+\frac{1}{2s}\right)\right]}$$

$$I(s) = \frac{s(s+2)\,V(s) \cdot 2s}{-(s+3)(4s^4+12s^3+9s^2+4s+1)}$$

$$\frac{I(s)}{V(s)} = \frac{2s^2(s+2)}{(s+3)(4s^4+12s^3+9s^2+4s+1)}$$

2-2-2 Mechanical Systems :-

Mechanical systems can be divided into two basic Systems.

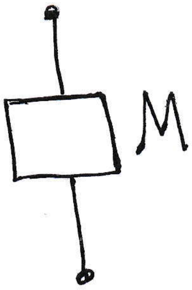
a) Translational systems b) Rotational Systems.

(a) Translational systems :-

1 - Mass

This represents an element which resists the motion due to inertia. According to Newton's second Law of motion, the inertia force is equal to mass times acceleration.

$$f_M = Ma = M\frac{dv}{dt} = M\frac{d^2x}{dt}$$



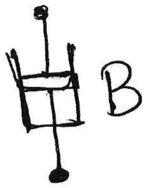
a = acceleration.

v = velocity.

x = displacement.

2- Dash pot

This is an element which opposes motion due to friction. If the friction is viscous friction, the fractional force is proportional to "velocity". This force is also known as damping force.



$$f_B = Bv = B\frac{dx}{dt}$$

where B is the damping coefficient. This element is Called as dash pot an

3- Spring

This third element which opposes motion is the Spring. The restoring force of a spring is propontional to the displacement.

$$f_K = K x$$

Where K is known as the stiffness of the spring or simply Spring Constant.

b) Rotational Systems

There are three basic elements representing rotational systems.

1- Moment of Inertia

This element opposes the rotational motion due to Moment of Inertia. The opposing inertia torque is given by

$$T_I = J a = J \frac{dw}{dt} = J \frac{d^2\theta}{dt^2}$$

a = angular acceleration.

w = angular velocity.

θ = angular displacement.

J = moment of inertia of the body.

2- Friction

The damping or frictional torque which opposes the rotational motion is given by,

$$T_B = B\omega = B\frac{d\theta}{dt}$$

B: is the rotational frictional coefficient.

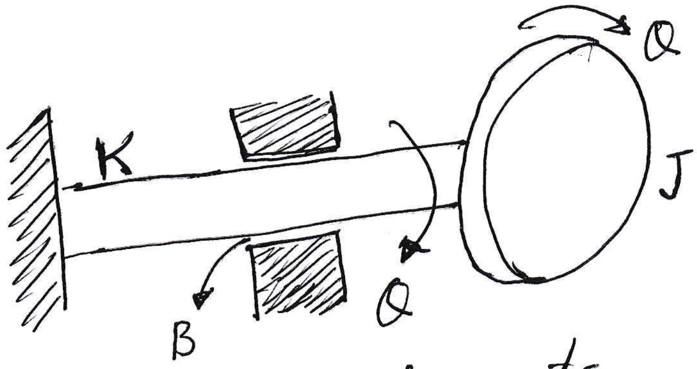
3- Spring

The resonting torque of a spring is proportional to the angular displacement θ and is given by

$$T_K = K\theta$$

Where K is the torsimal stiffness of the spring.

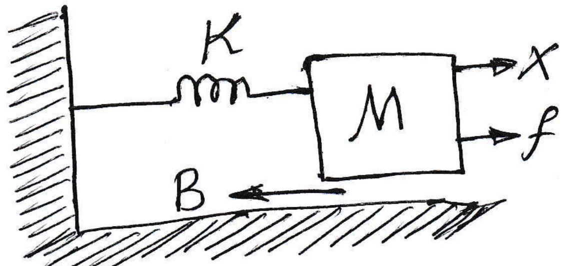
The three elements defined as shown in figure below.



Rotational elements

For any body, the algebric sum of externally forces and the forces opposing the motion in any given direction is zero.

— A mass M is fixed to a wall a spring K and the mass moves on the floor with a viscous friction. An external force f is applied to the mass. let us obtain the differential equation governing the motion of the body.



external force = f

resisting forces:-

$$\underbrace{M\frac{d^2x}{dt^2}}_{\text{inertia force}} + \underbrace{B\frac{dx}{dt}}_{\text{Damping force}} + \underbrace{Kx}_{\text{spring force}} = \underbrace{f}_{\text{external force}} \quad -(**)$$ By D'Alembert principle

$$Ms^2X(s) + BsX(s) + KX(s) = F(s)$$ By Laplace transform

$$X(s)\left[Ms^2 + Bs + K\right] = F(s)$$

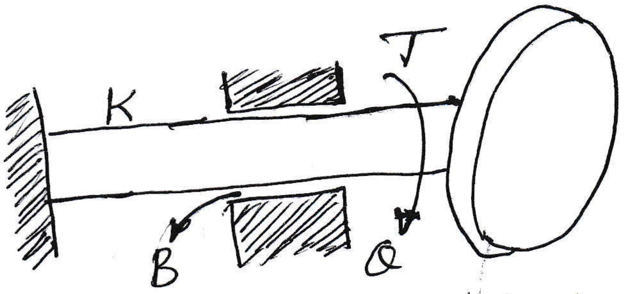
$$\frac{X(s)}{F(s)} = \frac{1}{Ms^2 + Bs + K}$$

If velocity is chosen as the output variable, we can write eq (**) as

$$M\frac{dv}{dt} + Bv + K\int v\,dt = f$$

— The differential equation governing the motion of rotational system can also be obtained for the system below.

$$J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} + K\theta = T$$



$$JS^2\theta(s) + BS\theta(s) + K\theta(s) = T(s)$$

$$\frac{\theta(s)}{T(s)} = \frac{1}{JS^2 + BS + K}$$